EuJAP | Vol. 10, No. 2, 2014
UDK:
1:51
510.2:1

MIŠČEVIĆ'S REPLY TO JIM, AKA JAMES ROBERT BROWN

NENAD MIŠČEVIĆ

ABSTRACT

Two topics dear to James Robert Brown are discussed, and brought together. First, the applicability of mathematics: it is claimed that applicability offers an a posteriori justification of our mathematical beliefs, on a reflective, rather holistic level in a two level hierarchy. Second, the answer to Benacerraf's dilemma. A non-empirical mathematical property M is realized in empirical reality through realizers, concrete numerical patterns. The realizers have been interacting causally with human thinkers throughout evolution, which has, through a kind of evolutionary abstraction process, left a proto-representation as of M in human mental apparatus. The innate proto-representation, and its ontogenetic avatars, guide actual humans in recognizing instances of M in the empirical reality. But does the evolutionary production of mechanisms for M-representation track truth? Hopefully yes; the applicability and indispensability of math are a testimony to the truth-tracking.

Keywords: *mathematics-applicability of, Benacerraf's dilemma, evolutionary account*

First apologies for the long delay in replying, to Jim and to other generous and patient authors!

I owe my interest in thought-experiments and intuitions to Jim. I listened to his talk(s), I decided to read some of his writing, and to do a little thinking myself. And I ended up hooked at the topic, for almost three decades. We spent some exciting time in Dubrovnik, when in the middle of the war we held a conference on thought-experiments. And we kept meeting in Dubrovnik and in Canada; Jim has been keeping the Dubrovnik conference on philosophy of science alive and well for more than two decades.

Let me now pass to the reply. I want briefly to address two topics. The first is the one of pure and applied math; I am honored by Jim's contributing to the volume with highly original ideas. The other is Benacerraf's dilemma that is in a subtle was central for Jim's platonic proposal. I want to use the occasion and briefly sketch my line of thought about the two topics in question. Let me start with the contrast between pure and applied mathematics. Here is Jim:

> The pure vs. applied distinction in ethics turns out to be similar to the mathematicians' pure vs applied distinction. Pure ethics and applied both cite concrete examples. Pure mathematics and applied — using the mathematicians' distinction, not the philosophers' — also both cite concrete examples. The difference between the pure and the applied is in our aims and interests, not in the examples. The ethics distinction parallels the mathematical. We distinguish which is which when we see how they are used.

I agree very much. But note, the view suggests a strong presence of mathematical reality within the physical one. If the math example and the physics example differ only relative to our „aims and interests", then either the apparently mathematical structure is nothing but a physical one, as nominalists would like to have it, or mathematical structure is strongly present in the physical one.[1] Jim and I share antipathy to nominalism in the philosophy of math; so, we are left with the strong presence alternative. I hope we agree that there are real (mathematical) features of the world, that elementary math captures some real (mathematical) features of the world and that many elementary mathematical intuitions are realistically correct (true).

Jim's examples are relevant here, especially his second one.

> One of the most interesting approaches in recent times is that of Alain Connes, a contemporary French mathematician (and Fields Medal winner in 1982). Connes (1996) used some of the machinery of quantum mechanics. He sets up an infinite dimensional Hilbert space to represent a quantum system, not of physical entities, but of prime numbers. The "energy levels" of the system are linked to the eigenvalues. So far Connes has shown that the energy levels correspond to non-trivial zeros of the zeta function and have real parts equal to ½. What is yet to be shown is that this includes all the non-trivial zeros. If yes, then the Riemann hypothesis would be true.

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[1] I am leaving aside some very sophisticated alternatives, like the „epistemic" one put forward by Christopher Pincock in se ries of papers and then in the Mathematics and Scientific Representation book (Oxford University Press, 2012), hoping to address them at some other occasion.

If prime numbers can be organized into a structure completely analogous to structure governing quantum systems, then the structural coincidence (all the way to isomorphism) is obvious. As Jim says:

> In the Riemann hypothesis example, physics, in the form of quantum mechanics does not motivate the problem (since the hypothesis is older than quantum theory); rather it guides us in a possible solution to the problem.

Let me point out possible connections of strong presence alternative in two areas, epistemology pure and epistemology connected with metaphysics.

The first has to do with applicability, and concerns the solution of a problem that we might call The Beautiful Mind Problem, that is related to the Cartesian doubt from madness. . The beginning of *Beautiful Mind* book "(Silvia Nasar, 1998, Simon & Schuster) reports Nash saying that he gets his intimation about extraterrestrials "from the very same source" from which he gets his mathematical intuitions (his statement is printed on the cover). A skeptic might appeal to this report and direct our attention to the phenomenology of armchair thought and claim the following. It is possible coherently to imagine states of irrational quasi-understanding, (quasi-) inferring and (quasi-) proof-following that are phenomenally indistinguishable from rational understanding, inferring and proof-following. Due to phenomenological indistinguishability, it is impossible to tell from the first-person perspective whether one is in such irrational state. Indeed, there is a fair chance that the actual pathology of thought exhibits such phenomena: some scenarios of such occasional attacks is not only possible, but seems to be actual. Therefore, even if one is de facto in the rational state, one cannot know this by reflection alone. Therefore one cannot in general know that one is undergoing an episode of rational understanding, rational inferring and proof-following, in contrast to their irrational counterparts.[2]

Now, applicability offers a solution. Nash's equilibrium theorems can be applied in economics, and have very successfully been so, whereas his info about extraterrestrials made no carrier at all. Successful application can justify pure mathematical speculation. This brings us close to the tradition of indispensability argumentation. Now, what is the right format for such considerations?

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[2] This is, of course, a variant of the argument from demon and madness, anticipated by Descartes, sketched by some of his interpreters, like, for instance H. Frankfurt and B. Williams, and developed by C. Wright, the Dreamers and Madmen Argument, as we might call it, borrowing from the title of Frankfurt's book.

My proposal is that it is the two level epistemology, proposed by authors like Ernst Sosa and John Greco.[3] The first level is immediate justification; they derive it from reliability, but I would add obviousness, especially for math. The second level is the reflective one, bringing together all other considerations that speak in favor of mathematical belief. Applicability as well as indispensability show their justificatory potential here: mathematical reasoning has proved to be massively successful, so why could not a reflective thinker appeal to this past success as one reason for her trust? This widens the basis for legitimating the belief, bringing in some a posteriori elements. Our philosopher is not thereby becoming a fanatical Quinean. She can, in defiance to reliability, retain the obviousness and compellingness as her immediate reason. But if questioned further, about credential of such immediate obviousness, she might reply that it has not deceived her ever, and has been empirically successful. So, these more holistic, success-involving considerations might be taken in the usual way to indicate objective reliability of one's methods. This is the way realism functions in science and in commonsense. A good, probably the best, explanation of success is truth(-likeness) and reliability. Logic is, of course used in the process of integration, and is at the same time justified by its role in it; it integrates itself by helping the thinker to integrate the whole of one's beliefs. But the circle is virtuous, not vicious, the alleged circularity being the usual circularity of holistic, coherentist justification.

Notice the symmetrical opposite case: Perceptual beliefs get integrated with the rest with the help of massive use of deductive logic and (often, but not always) inductive schemas which are to a large extent a priori. So, the empirical gets mixed with the a priori, the same way in which the armchair gets integrated with the empirical.

So much for the importance of applied mathematics for epistemology of mathematics.

Let me pass to the second issue, our contact with mathematical reality. How do we access mathematical items ("MIs" for short) ? For instance, concrete and abstract patterns, arithmetical and geometrical, then numerical properties (cardinality, ordinal position), numbers themselves, sets and the like? Benacerraff's dilemma has been the main puzzle in the area. Jim answers it by appeal to analogy with the perceptual knowledge:

[3] See:

Greco, J., 2010: Achieving Knowledge, A Virtue-Theoretic Account of Epistemic Normativity, Cambridge: Cambridge University Press.

Sosa, E., 2007: A Virtue Epistemology, Apt Belief and Reflective Knowledge Vol. 1, Oxford: Oxford University Press.

Sosa E., 2009: Reflective Knowledge, Apt Belief and Reflective Knowledge Vol. 2, Oxford: Oxford University Press.

Sosa, E., 2011: Knowing full well, Princeton, Princeton: Princeton University Press

> Benacerraf remarks that the nature of abstract objects ‘places them beyond the reach of the better understood means of human cognition (e.g., sense perception and the like)’ (1973, 409). But how much more do we know about physical perception than mathematical intuition? In the case of ordinary visual perception of, say, a teacup, we believe that photons come from the physical teacup in front of us, enter our eye, interact with the retinal receptors and a chain of neural connections through the visual pathway to the visual cortex. After that we know virtually nothing about how beliefs are formed. The connection between mind and brain is one of the great problems of philosophy. Of course, there are some sketchy conjectures, but it would be completely misleading to suggest that this is in any way ‘understood’. Part of the process of cognition is well understood; but there remain elements which are just as mysterious as anything the platonist has to offer. *The Laboratory of the Mind: Thought Experiments in the Natural Science*s, Routledge, 1993 p. 65.

And he notes that even the simplest perceptual connections have something mysterious about them.

> Let’s face it: we simply do not know how the chain of physical events culminates in the belief that the teacup is full. Of course, we should not glory in this state of ignorance. I suggest only that mathematical intuition is no more mysterious than the final link in physical perception. We understand neither; perhaps some day we will understand both. (Ibid.)

However, the puzzle remains. My proposal, very brief one is that we intuit the concrete MIs (instantiated patterns, pairs, trios, quartets of things of all sorts) and we reason to the more abstract ones. Assume that there are real (mathematical) features of the world. Then, elementary math captures some real (mathematical) features of the world. So, I want to build my account by developing and correcting the extant pattern-recognition based epistemological proposals, and detaching them from narrowly structuralist ontology:

Here are the steps: first, mathematical items, MIs (properties, structures, etc.) are instantiated in nature. (The view formulated by classics like Aristotle (Met. M,3, see the Oxford edition 1976 and comments by Julia Annas) and nicely accounted for by structuralists. Second, instantiated MIs play a role in explanation, therefore, instantiated MIs can explain our (initial) knowledge of mathematical objects. Consider first the instantiation: Stewart Shapiro has put the point nicely stressing the structure as the common denominator:

> "My account of the relationship between mathematics and science begins with the suggestion that the contents of the non-mathematical universe exhibit underlying mathematical structure in their interrelations and interactions. For example, it might be

> claimed that a mathematical structure similar to the inverse-square variation of real numbers is exemplified in the mutual attraction of physical objects. In general, physical laws expressed in mathematical terms can be construed as proposals that a certain mathematically defined structure is exemplified in a particular area of physical reality" (Mathematics and Reality, Phil. of Science, dec. 1983., p. 538).

Partly innate competence have originally developed in ordinary interaction with ordinary objects, and they are being honed in practice. For instance, once you have a series representing numerosity, the manipulation of such a series can illustrate simple numerical operations. It makes obvious simple arithmetical truths, or at least their applied variants, like for instance, that 3 strokes added to 2 strokes make 5 strokes. Indeed putting together a three series, "III", and a two series ,"II", yields a five series: "IIIII The contents of such intuitions seem to be concrete instances of mathematical structures, in this case of natural number. (See Parsons *Mathematics in Philosophy*, Cornell UP, 1983.p.43 ff.) We do intuit the concrete mathematical items (instantiated patterns, pairs, trios, quartets of things of all sorts), and, as Frege would put it seeing what number is as if through a mist. How is this possible?

The problem is simple to state but quite deep. Of course, we do perceive objects and our perception concentrates upon similarities in color, overall shape and Gestalt-organization, and, when faced with a choice, disregards more abstract similarities like the numerical one. Therefore, as against the structuralist mainstream, Parsons and Resnik in particular, no purely perceptual device would do for recognizing instantiated (structural) mathematical properties.

Typically perceptual properties like shape, color, gestalt-features in general are manifestly unsuitable. The conceptualist critics then often suggest that the number concept is the only one which would. Intuition would then be demoted from its role of providing the first contact with numerical properties. We hope to have found a middle road, with the hint from the cognitive research: simple, pre-conceptual, or, at most proto-conceptual operations are sufficient to ensure the right initial classification (typing) of particular items into numerically relevant types. Further, the operations might at the outset be performed by purely unconscious, mechanical, image-based or proto-computational routines. Their initial result might be that the beginner (a child) treats sequences as if they were classified by their proto-numerical properties. Later on, part of the procedure--or even the procedure as a whole might become conscious, r even an object of reflective attention and deliberation, but retaining the same abstract structure, the very scaffolding which filters out the right properties. we reason to the non-instantiated ones. We thus have indirect contact with “original” mathematical universals, and a naturalistically explainable competence.

Note that the definition of equinumerosity in terms of matching (1-1 correlation) suggests that one can establish it by imaginatively matching the elements of two or more collections. Now if basic imaginative intuitions are really fundamental for acquiring numerical knowledge, and if that knowledge does concern abstract properties, it follows that imagination can reveal abstract properties. If this holds, an account based on such assumption might (and I hope would) explain in a non-question begging way how the abstractness of the target properties is compatible with their quasi-perceptual detection. But how can abstract numerical properties be thus detected?

Let me start with a typical quandary. A child can figure out that there as many eggs in the refrigerator as there are fingers on her hand. But "how many" question has no definite answer for simple collections or aggregates, Frege has taught us. Later we started believing that we need sets in order to talk about the same number of things. Casullo has nicely put the point many years ago:

> "… from the epistemic point of view, it is essential that number be a property of aggregates of spatio-temporal particulars such as apples. On the other hand, when we turn to the semantic requirements of a plausible analysis of mathematical propositions, number must be treated as a property of sets" (Casullo, A. "Causality, Reliabilism and Mathematical Knowledge," Philosophy and Phenomenological Research, sept.1992,v.LII NO.3, p. 565.)

The five eggs in my refrigerator are not a set in my refrigerator, indeed. However, a plurality of objects already identified as eggs, comes close to instantiating a set. Since the child normally perceives the objects as apples, rather than as collections of disparate parts, she has at her disposal the minimal means to correlate the objects with her fingers, again perceived and thought of as being a single object each.

Structuralism offers us a plausible metaphysics to be combined with the epistemic message from cognitive science. In a nutshell, the idea is that the initial degree of abstractness is achieved through the choice of the right operation: being insensitive to many concrete properties--color, size, and the like--the matching detects the abstract structural property of (equi-)numerosity. This goes a long way towards solving the problem of 'perceiving' concrete objects as instances of abstract number-structure. Of course, the proposal is also compatible with less Platonic views than the ones of mainstream structuralists, like Parsons and Shapiro; it is compatible with Neo-aristotelian views (such as, for instance James Franklin's from his *Philosophy of Mathematics-Mathematics as the Science of Quantity and Structure*, Palgrave Macmillan, 2014), and with even more deflationary views; however its strength lies in its compatibility with ambitious structuralist Platonist proposals.

We have innate mental-neural mechanisms (selected for) recognizing proto-mathematical relations, such as one-one matchability; they have developed in our evolutionary history (apparently some birds can match small collections of objects) for obvious advantages they bring. This gives the "magical ingredient" that has to be added to perception in order to yield the capacity to recognize numerical properties. There are simple imaginative or quasi-perceptual operations which detect such proto-numerosity and enable the naïve cognizer to use concrete sequences as representatives of their abstract types, e.g. a s2equence of strokes as an instance-example of its (mathematical) sequence-type. The detection is not mysterious. The abstract property, numerosity, is represented by the relational property of matchability, and matchability can be detected by actual matching.

However a critic might attack us at this juncture (I have had two of them, Majda Trobok and my student Zsolt Novak, (who found inspiration in Balaguer,(1998), Platonism and anti-platonism in mathematics, Oxford University Press) and I thank both. She or he might represent the reasoning in question in a Frege-like way:

> Since apples and fingers are 1-1 related, their number is the same.

But, the antecedent here is concrete, whereas the consequent introduces an abstract platonic object, number. The child can see that the antecedent holds, not that the consequent does.

To see who wins, we need an understanding of abstractness. Alternative notions of abstractness have been central to philosophy since the earliest records of the discipline too. In the current literature, there are at least three notions that must be clearly distinguished from each other. The first can be contrasted with the notion of concreteness, the second with the idea of particularity, while the third with the concept of spatiotemporality.

In the first sense of the term, something is abstract if and only if it can exist merely as an aspect of a concrete entity. Autonomous existence, some philosophers say, is the privilege of concrete entities. On this construal, properties of concrete individuals, such as the colour of this rose in front of me, count as abstract entities, while concrete objects that can exist only outside space and time, such as the number one, do not.

In the second sense of the term, something is abstract if and only if it can be fully present at various spatiotemporal locations. On this understanding, universal properties, such as the property of redness in general, are abstract entities, while the particular instances of this property or the particular objects of a non-spatiotemporal domain are not.

Finally, in the third sense of the term, something is abstract if and only if its existence is atemporal (i.e. it exists, if at all, outside the natural world).

Adopting this construal, atemporal objects and properties, such as numbers, ante rem structures, the content of Kantian categorical imperatives, moral values, Fregean senses, propositions, inferential relations and truth values, are abstract entities, while universal properties instantiated in space and time, or the particular instances themselves, are not. (I am using formulations that Zsolt proposed in his dissertation).

Zsolt noted that the subject matter of a thought or sentence can be abstract without being platonic in character. Directions, for instance, as universal properties can characterise fictive and real spatiotemporal objects as well. An account of our ability to refer to and acquire knowledge of abstract entities may therefore amount to a platonist response to Benacerraf's challenge only if the subject matter of this knowledge is abstract in the required sense of the term. In contrast to the numerical terms of pure mathematics, however, the denoting expressions appearing on the left hand side of the instances of Frege's Abstraction Principle do not necessarily stand for abstract entities in the required sense of the term (i.e. they do not necessarily stand for strictly non-spatiotemporal entities).

Let us apply this to our example. The expression 'the number of apples in front of me', for instance, primarily stands for a property of a group of objects in the spatiotemporal world, rather than for an object of a platonic realm. Other denoting phrases, such as the expression 'the number of primes between 70 and 80', refer to a property of non-spatiotemporal entities, but the contextual definitions provided by Hume's Principle in these examples presuppose that we already acquired some concepts and knowledge of a non2spatiotemporal domain. Putting it briefly, what the account seems to explain is how we can develop new concepts of certain fields from some earlier acquired ones of the very same fields. What it fails to explain is how we develop our notions of entities that cannot appear in space and time in the first place, maybe relying on our notions acquired earlier of entities appearing in space and time.

Let me appeal to the idea of degrees of concreteness connected to the three senses of "abstract": the child sees that the two pluralities have something, a pattern, in common, and this is abstractness in the first, and perhaps second sense. Later, she can use the number-word as attribute for describing pluralities (of previously identified particulars), and this is not totally misleading; indeed, Russell held the view that number is an attribute, not a singular term. Next, at some adult stage, one has first, a number word as a name for abstract object, then, if one reflects a lot, a proto-theory about it.

Number is not abstracted from things in ways colors,…, are, Frege famously said in his *Foundations*, § 45. I agree, it is arrived at indirectly, through concrete pluralities. Pattern-recognition story is not a non-contact account, if we count indirect contact as contact (which we do: contact through gloves, boxing gloves, and even through skype).

On any reasonable pattern-recognition story we come in touch with Platonic mathematical objects indirectly:one has first, direct (perceptual, and quasi-perceptual intuitional contact with instantiated MIs, then indirect contact with their abstract “originals” (at late, proto-theoretical stage). This is the way up through thorns to the platonic stars.

And to conclude, I would have never written about such topics had it not been Jim's early and lasting influence! So again, thanks a lot!